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AXISYMMETRIC VORTEX FLUID FLOW IN A LONG ELASTIC TUBE

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A mathematical model for axisymmetric eddy motion of a perfect incompressible fluid in a long tube with thin elastic walls is proposed. Necessary and sufficient conditions for hyperbolicity of the system of equations of motion for flows with monotonic radial velocity profiles are formulated. The propagation velocities of the characteristics of the system under study and the characteristic shape of this system are calculated. The existence of simple waves continuously attached to a given steady shear flow is proved. The group of transformations admitted by the system is found, and submodels that determine invariant solutions are given. By integrating factor-systems, new classes of exact solutions of equations of motion are found.

In [1, 2] and some other studies on hydrodynamics of large blood vessels, a fluid flow in thin-walled elastic tubes was considered. In these works, the predominant part of analytical solutions that describe propagation of disturbances in the fluid and characteristic modes of such a flow was obtained either in the linear approximation or for one-dimensional flows. In the present study, a nonlinear model of an axisymmetric vortex flow in a long cylindrical tube with elastic isotropic walls is considered. On the basis of the method of generalized characteristics [3] proposed for systems with operator coefficients, the propagation velocities of the characteristics are determined, and conditions for hyperbolicity of the equations are found. The existence of solutions of the model under study in the class of simple waves is established. With the use of the group-analysis methods for differential equations [4], classes of exact solutions of the equations of motion are constructed.

1. Derivation of the Mathematical Model. The flow of an incompressible fluid under the assumption of axial symmetry of this flow is described, in dimensionless variables, by the Euler equations

$$U_t + UU_x + VU_r + p_x = 0, \quad \varepsilon^2(V_t + UV_x + VV_r) + p_r = 0, \quad U_x + V_r + V/r = 0. \tag{1.1}$$

Here $T = L_0U_0^{-1}t$, $X = L_0x$, $r_1 = R_0r$, $U_1 = U_0U$, $V_1 = R_0L_0^{-1}U_0V$, and $p_1 = \rho U_0^2 p$ are the time, axial and radial coordinates, velocity components, and pressure, t, x, r, U, V, and p are the corresponding dimensionless variables, L_0 is the characteristic scale along the X-axis (tube axis), R_0 is the inner radius of the tube at zero excess pressure, and ρ is the density of the fluid; the quantity U_0 has the dimension of velocity. The fluid is assumed to occupy the whole flow region (the interior of the tube). In the case of a long cylindrical tube, the parameter $\varepsilon = R_0L_0^{-1}$ is small, and the terms of order ε^2 in Eqs. (1.1) may be neglected. Then, the second equation in (1.1) reduces to $p_r = 0$; hence, under an excess pressure, the deformed tube retains its cylindrical shape. This allows us to write the boundary conditions in the form

$$R_t + U(t,x,R)R_x = V(t,x,R), \qquad V(t,x,0) = 0, \tag{1.2}$$

where $R = R_1R_0^{-1}$ [$R_1(t,x)$ is the dimensional radius of the deformed tube]. Equations (1.1) (with $\varepsilon = 0$) and (1.2) form a nonclosed system of equations for the following dimensionless quantities: velocities $U(t,x,r)$ and $V(t,x,r)$, pressure $p(t,x)$, and inner radius of the tube $R(t,x)$ in the region $-\infty < x < \infty$, $0 \leqslant r \leqslant R$ for $t \geqslant 0$.

To close the equations of motion, one should set a relation between the deformation of the elastic tube and the excess pressure in it. In the simplest case of small deformations, Hooke's law may by used. However, as is

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shown in [2], the form of Hooke's law to be used depends on the problem to be solved and should be different for different rates of the change in the cross-sectional area of the tube. In the literature (see [1, 2]), the following forms of Hooke's law were used: $p_1 = C(S-1) + p_0$, $p_1 = C(1 - 1/S) + p_0$, $p_1 = C \ln S + p_0$, etc. Here $S(t,x) = R^2$ is the ratio between the cross-sectional area of the deformed tube and that of the nondeformed tube, $C = E\delta/d$ is the wall elasticity, E is the Young modulus, d is the mean diameter, and δ is the tube thickness. In the first order of expansion in the Taylor series, all the above formulas are identical. We assume that the relation between the dimensionless pressure and the squared radius is given by a certain doubly differentiable relation $p = p(R^2/2)$ with $p' > 0$.

Using the substitution of variables [5]

$$y = r^2/2, \qquad u = U, \qquad v = rV, \qquad h = R^2/2, \tag{1.3}$$

we can reduce the problem of interest to studying plane–parallel flows of a perfect fluid in a channel with a rigid smooth bottom wall $y = 0$ and an elastic upper wall $y = h(t,x)$. Indeed, in terms of new variables, Eqs. (1.1) (with $\varepsilon = 0$) and (1.2) acquire the form

$$\begin{gathered} u_t + uu_x + vu_y + p_x = 0, \qquad u_x + v_y = 0, \qquad p = p(h), \\ h_t + u(t,x,h)h_x = v(t,x,h), \qquad v(t,x,0) = 0. \end{gathered} \tag{1.4}$$

It should be noted that, in the case of an elongated channel, the vorticity ω is proportional to u_y. With zero vorticity, system (1.4) describes one-dimensional flows in an elastic tube [1]. In a particular case with $p = h$, Eqs. (1.4) coincides with the well-known eddying shallow water equations [6]. Plane–parallel eddying flows of a fluid with a free boundary and such flows in a channel with rigid stationary walls were considered in [5–13] and some other works. Thus, in view of (1.3), the description of axisymmetric eddying fluid flows in an elongated elastic tube reduces to determining the functions $u(t,x,y)$, $v(t,x,y)$, and $h(t,x)$ entering system (1.4).

In some cases, the equations of motion can be conveniently analyzed in an Eulerian–Lagrangian coordinate system obtained by the substitution of the variable [7]

$$y = \Phi(t,x,\lambda) \qquad (0 \leqslant \lambda \leqslant 1), \tag{1.5}$$

where the function $\Phi(t,x,\lambda)$ is the solution of the Cauchy problem

$$\Phi_t + u(t,x,\Phi)\Phi_x = v(t,x,\Phi), \qquad \Phi(0,x,\lambda) = \Phi_0(x,\lambda) \tag{1.6}$$

with $\Phi(t,x,0) = 0$ and $\Phi(t,x,1) = h(t,x)$. Substitution (1.5) is reversible if $\Phi_\lambda \neq 0$. Next, we assume that $\Phi_\lambda > 0$. From Eqs. (1.4) and (1.6), we obtain the following system for the functions $u(t,x,\lambda)$ and $H(t,x,\lambda) = \Phi_\lambda(t,x,\lambda)$

$$u_t + uu_x + p'\Big(\int_0^1 H\,d\lambda\Big)\int_0^1 H_x\,d\lambda = 0, \qquad H_t + (uH)_x = 0. \tag{1.7}$$

If the functions u and H are found, then the upper boundary of the channel, $h(t,x) = \int_0^1 H\,d\lambda$, is known. The formulas $p = p(h)$, $\Phi_\lambda = H$, and $\Phi(t,x,0) = 0$, together with formulas (1.6), permit determination of the pressure, the function Φ, and the vertical velocity v.

2. Conditions for Hyperbolicity of the Equations of Motion. Equations (1.7) may be written as

$$\boldsymbol{U}_t + A\boldsymbol{U}_x = 0, \tag{2.1}$$

where $\boldsymbol{U} = (u,H)^{\mathrm{t}}$ and A is a matrix with operator coefficients that acts on an arbitrary vector-function $\boldsymbol{\varphi} = (\varphi_1,\varphi_2)^{\mathrm{t}}$ as follows:

$$A\boldsymbol{\varphi} = \Big(u\varphi_1 + p'\Big(\int_0^1 H\,d\lambda\Big)\int_0^1 \varphi_2\,d\lambda,\ H\varphi_1 + u\varphi_2\Big)^{\mathrm{t}}.$$

According to [3], the characteristic curve of system (2.1) is given by the differential equation $x'(t) = k(t,x)$, where k is the eigenvalue of the operator A^* (propagation velocity of the characteristic). The solution of the equation

$$(\boldsymbol{F}, (A - kI)\boldsymbol{\varphi}) = 0 \tag{2.2}$$

relative to the vector functional $\boldsymbol{F} = (F_1, F_2)$ is sought in the class of locally integrable or generalized functions. The action of the functional $\boldsymbol{F}$ is exerted through the variable λ (t and x are considered as parameters), and I is the unitary matrix. As a result of the action of the functional $\boldsymbol{F}$ on Eq. (2.1), we obtain the following relation in the characteristic:

$$(\boldsymbol{F}, \boldsymbol{U}_t + k\boldsymbol{U}_x) = 0. \tag{2.3}$$

System (2.1) is a hyperbolic one if all eigenvalues k are real and the set of relations on characteristics (2.3) is equivalent to Eqs. (2.1).

In its structure, system (1.7) is close to an integral–differential system that describes, in Eulerian–Lagrangian coordinates, plane–parallel eddy flows of a perfect barotropic fluid with a free boundary. For the barotropic-fluid model, the propagation velocity of the characteristics, the characteristic form of the system, and the conditions for hyperbolicity were obtained in [8]. In view of this, intermediate calculations here are omitted.

It can be easily shown that nontrivial solutions of Eq. (2.2) exist only for $k = k_j$ which satisfy the characteristic equation

$$\chi(k) = 1 - p'\Big(\int_0^1 H\,d\lambda\Big)\int_0^1 \frac{H\,d\lambda}{(u-k)^2} = 0, \tag{2.4}$$

and for the values of $k = k^\lambda$ that belong to the range of the function $u(t, x, \lambda)$. For flows with monotonic radial velocity profiles $u_\lambda \neq 0$ (for definiteness, we assume here that $u_\lambda > 0$), Eq. (2.4) has two real roots: $k_1 < u_0 = u(t, x, 0)$ and $k_2 > u_1 = u(t, x, 1)$. The latter follows from the fact that $\chi(k) \to 1$ as $|k| \to \infty$ and $\chi(k) \to -\infty$ as $k \to u_0$ and $k \to u_1$ ($p' > 0$). In addition, $\chi'(k) < 0$ if $k < u_0$, and $\chi'(k) > 0$ if $k > u_1$. The functionals corresponding to the eigenvalues k_1, k_2, and k^λ coincide with those obtained in [8] and have the form

$$(\boldsymbol{F}^j, \boldsymbol{\varphi}) = \int_0^1 \frac{H\varphi_1}{(u-k_j)^2}\,d\lambda - \int_0^1 \frac{\varphi_2}{u-k_j}\,d\lambda \quad (j = 1, 2), \quad (\boldsymbol{F}^{\lambda 1}, \boldsymbol{\varphi}) = -\varphi_1'(\lambda) + u_\lambda H^{-1}\varphi_2(\lambda),$$

$$(\boldsymbol{F}^{\lambda 2}, \boldsymbol{\varphi}) = \varphi_1(\lambda) + p'\Big(\int_0^1 H\,d\nu\Big)\int_0^1 \frac{H(\nu)(\varphi(\nu) - \varphi(\lambda))}{(u(\nu) - u(\lambda))^2}\,d\nu - \int_0^1 \frac{\varphi(\nu)}{u(\nu) - u(\lambda)}\,d\nu$$

(the integrals here are to be taken in the sense of the main value and the arguments t and x of the functions are omitted for brevity).

To reduce system (1.7) to its characteristic form, let us act on the equations with the eigenfunctionals F^j ($j = 1$, 2, $\lambda 1$, $\lambda 2$). After some manipulations, we obtain

$$\begin{gathered}
\int_0^1 \frac{H(\nu)(u_t(\nu) + k_j u_x(\nu))}{(u(\lambda) - k_j)^2}\,d\nu - \int_0^1 \frac{H_t(\nu) + k_j H_x(\nu)}{u(\lambda) - k_j}\,d\nu = 0, \\
u_{\lambda t} + u u_{\lambda x} - u_\lambda H^{-1}(H_t + u H_x) = 0, \\
u_t + u u_x + p'\Big(\int_0^1 H\,d\lambda\Big)\int_0^1 \frac{H(\nu)(u_t(\nu) + u(\lambda)u_x(\nu) - u_t(\lambda) - u(\lambda)u_x(\lambda))}{(u(\lambda) - u(\nu))^2}\,d\nu \\
- p'\Big(\int_0^1 H\,d\lambda\Big)\int_0^1 \frac{H_t(\nu) + u(\lambda)H_x(\nu)}{u(\lambda) - u(\nu)}\,d\nu = 0.
\end{gathered} \tag{2.5}$$

The conditions for hyperbolicity of Eqs. (1.7) are formulated in terms of the complex functions $\chi^\pm(t, x, \lambda)$, which are ultimate values of the analytical function $\chi(z)$ in the upper and lower half-planes in the segment $[u_0, u_1]$:

$$\chi^\pm = 1 + p'(h)\Big(\frac{1}{\omega_1(u_1 - u(\lambda))} - \frac{1}{\omega_0(u_0 - u(\lambda))} - \int_0^1 \Big(\frac{1}{\omega}\Big)_\nu \frac{d\nu}{u(\nu) - u(\lambda)} + \pi i\Big(\frac{1}{\omega}\Big)_\lambda \frac{1}{u_\lambda}\Big).$$

Here $\omega = u_\lambda H^{-1}$, $\omega_0 = \omega(t, x, 0)$, $\omega_1 = \omega(t, x, 1)$, and i is the imaginary unit.

Theorem 1. *Let* u and ω *be differentiable functions,* u_λ, ω_λ, and H *satisfy the Hölder condition for the variable* λ ($u_\lambda \neq 0$ and $\chi^+ \neq 0$). *Then, the conditions*

$$\chi^{\pm}(\lambda) \neq 0, \qquad \Delta \arg(\chi^+(\lambda)/\chi^-(\lambda)) = 0 \tag{2.6}$$

(*the argument increments here are calculated for* λ *varying from* 0 *to* 1) *are necessary and sufficient ones for hyperbolicity of Eqs.* (1.7).

The proof of this theorem is analogous to that reported by Teshukov [8]. Thus, in the region of their hyperbolicity, Eqs. (1.7) are equivalent to relations on the characteristics given by (2.5).

3. Existence of Simple Waves. Simple waves are understood as the solutions of Eqs. (1.7) of the following form:

$$u = u(\alpha(t,x),\lambda), \qquad H = H(\alpha(t,x),\lambda). \tag{3.1}$$

Such solutions for the eddying shallow water equations were considered by Teshukov [9]. According to (3.1), the functions $u(\alpha,\lambda)$ and $H(\alpha,\lambda)$ can be found from the system

$$(u-k)u_\alpha + p'\Big(\int\limits_0^1 H\,d\lambda\Big)\int\limits_0^1 H_\alpha\,d\lambda = 0, \qquad (u-k)H_\alpha + Hu_\alpha = 0, \qquad k = -\frac{\alpha_t}{\alpha_x}. \tag{3.2}$$

It follows from formula (2.1) that the simple waves described by Eqs. (1.7) are solutions of the system $(A - kI)\boldsymbol{U}_\alpha = 0$, and the relations on the characteristics acquire the form $(\boldsymbol{F}^j, (A-kI)\boldsymbol{U}_\alpha) = (k_j - k)(\boldsymbol{F}^j, \boldsymbol{U}_\alpha) = 0$ ($j = 1, 2, \lambda 1, \lambda 2$). In the region of hyperbolicity, Eqs. (1.7) are equivalent to the relations on the characteristics; hence, nontrivial solutions in the class of simple waves exist only if k coincides with one of the roots of the characteristic equation (2.4) [$k = k_1(\alpha)$ or $k = k_2(\alpha)$] or belongs to the segment of the function u [$k = u(\alpha,\lambda(\alpha))$]. In the present work, the existence of simple waves for a discrete characteristic spectrum is proved. Let us consider the case $k = k_2(\alpha)$ ($k_2 > u$). As the variable α, we take the function $h(t,x) = \int\limits_0^1 H\,d\lambda$.

The characteristics of a simple wave propagate at constant velocities $x'(t) = k$, and in the space of variables (t,x,λ), the domain of definition of the simple wave is covered by the family of planes $h = \mathrm{const}$ within which the functions u and H depend only on the variable λ. We consider the problem of attachment of a simple wave to a given shear flow $u = u_0(\lambda)$ and $H = H_0(\lambda)$ along the characteristic $h = h_0 = \int\limits_0^1 H_0(\lambda)\,d\lambda$. From system (3.2) and Eq. (2.4) for the functions $u(h,\lambda)$, $H(h,\lambda)$, $u_\lambda(h,\lambda)$, and $k(h)$, we obtain the problem

$$u_h = -\frac{p'(h)}{u-k}, \qquad H_h = \frac{p'(h)H}{(u-k)^2}, \qquad u_{\lambda h} = \frac{p'(h)u_\lambda}{(u-k)^2},$$

$$k_h = -\Big(\frac{3p'(h)}{2}\int\limits_0^1 \frac{H\,d\lambda}{(u-k)^4} + \frac{p''(h)}{(p'(h))^2}\Big)\Big(\int\limits_0^1 \frac{H\,d\lambda}{(u-k)^3}\Big)^{-1}, \tag{3.3}$$

$$u|_{h=h_0} = u_0(\lambda), \quad H|_{h=h_0} = H_0(\lambda), \quad u_\lambda|_{h=h_0} = u_0'(\lambda), \quad k|_{h=h_0} = k_2^0,$$

where k_2^0 is the root of the equation $p'(h)\int\limits_0^1 (u_0(\lambda)-k)^{-2}H(\lambda)\,d\lambda = 1$ such that $k_2^0 > u_0(\lambda)$.

The following property of simple waves is noteworthy: if $u(h_0,\lambda_1) \neq u(h_0,\lambda_2)$, then $u(h,\lambda_1) \neq u(h,\lambda_2)$ everywhere in the domain of definition. The latter follows from the uniformity of the equation with respect to the difference $s(h) = u(h,\lambda_1) - u(h,\lambda_2)$:

$$s_h = \frac{p'(h)}{(u(h,\lambda_1)-k(h))(u(h,\lambda_2)-k(h))}\,s(h), \qquad s(h_0) = 0.$$

Therefore, it suffices that the monotonicity of the velocity profile at $h = h_0$ be demanded.

Theorem 2. *Let* $u_0(\lambda)$ *be a continuously differentiable function and* $H_0(\lambda)$ *be a function continuous on the segment* $[0,1]$*, such that*

$$u_0'(\lambda) > 0, \quad H_0(\lambda) > \delta > 0, \quad k_2^0 > u_0(1) + \delta, \quad (\omega_0(\lambda))^{-1} = H_0(\lambda)(u_0'(\lambda))^{-1} \geqslant a > 0, \tag{3.4}$$

and conditions (2.6) *are fulfilled. Then, system* (3.3) *has a unique solution in each interval* $h \in (0,b]$ ($h_0 \in (0,b]$), *and* $u(h,\lambda)$ and $H(h,\lambda)$ *are a differentiable and a continuous functions, respectively.*

Proof. We introduce the Banach space B of elements u, u_λ, H, and k with the norm

$$\|\boldsymbol{V}\| = \max_\lambda |u| + \max_\lambda |u_\lambda| + \max_\lambda |H| + |k|.$$

Equations (3.3) may be written in the vector form

$$\frac{d\boldsymbol{V}}{dh} = \boldsymbol{G}(\boldsymbol{V}), \qquad \boldsymbol{V}(h_0) = \boldsymbol{V}_0 \tag{3.5}$$

[$\boldsymbol{G}(\boldsymbol{V})$ is an operator in the space B]. We make use of the following well-known theorem [14]: if there exists $\varepsilon > 0$ such that at $\|\boldsymbol{V} - \boldsymbol{V}_0\| < \varepsilon$, and the inequalities

$$\|\boldsymbol{G}(\boldsymbol{V})\| \leqslant M, \qquad \|\boldsymbol{G}(\boldsymbol{V}_1) - \boldsymbol{G}(\boldsymbol{V}_2)\| \leqslant N\|\boldsymbol{V}_1 - \boldsymbol{V}_2\| \tag{3.6}$$

hold, then problem (3.5) for $|h - h_0| < \varepsilon_1 = \min(\varepsilon M^{-1}, N^{-1})$ has a unique solution $\boldsymbol{V}(h) \in B$ such that $\|\boldsymbol{V} - \boldsymbol{V}_0\| \leqslant \varepsilon$.

Let us check that the conditions of the above-formulated theorem are fulfilled. For all elements of the space B from the ball $\|\boldsymbol{V} - \boldsymbol{V}_0\| < \delta/2$, in view of (3.4), the inequalities $|u - k| \geqslant |u_0 - k_0| - |u - u_0 + k - k_0| > \delta/2$ and $|H| \geqslant |H_0| - |H - H_0| > \delta/2$ hold. Since the functions that define the nonlinear mapping $\boldsymbol{G}(\boldsymbol{V})$ for $|u - k| > \delta/2$ and $|H| > \delta/2$ are continuous, there are constants M and N that depend on δ and $\|\boldsymbol{V}_0\|$, for which inequalities (3.6) are valid; hence, there exists a unique solution of Eqs. (3.5) for $|h - h_0| < \varepsilon_1$. Thus, the local theorem is proved.

To prove the existence of simple waves for all h, the following *a priori* estimates are used:

$$\frac{a\sqrt{h}p'(h)}{a\sqrt{p'(h)} + \sqrt{h}} \leqslant k - u \leqslant \frac{h}{a} + \sqrt{hp'(h)}. \tag{3.7}$$

It follows from the second equation of system (2.5) that, in the region with a simple wave, the vorticity is $\omega = \omega_0(\lambda) = u_0'(\lambda)/H_0(\lambda)$; hence, $\omega^{-1} = \omega_0^{-1} \geqslant a > 0$. The inequalities

$$0 < k - u_1 \leqslant k - u \leqslant k - u_0, \tag{3.8}$$

Eq. (2.4), and the expression $h = \int\limits_0^1 H\,d\lambda = \int\limits_{u_0}^{u_1} \omega^{-1}\,du$ allow the following estimates to be obtained:

$$k - u_0 \geqslant \sqrt{hp'(h)}, \qquad k - u_1 \leqslant \sqrt{hp'(h)}, \qquad u_1 - u_0 \leqslant a^{-1}h. \tag{3.9}$$

From relation (2.4), we have

$$1 = p' \int\limits_0^1 \frac{H\,d\lambda}{(u-k)^2} = p' \int\limits_{u_0}^{u_1} \frac{du}{\omega(u-k)^2} \geqslant ap' \int\limits_{u_0}^{u_1} \frac{du}{(u-k)^2} = \Big(\frac{1}{u_0 - k} - \frac{1}{u_1 - k}\Big) ap'. \tag{3.10}$$

From (3.9) and (3.10), it follows that

$$\begin{gathered} k - u_1 \geqslant \frac{ap'(h)(k - u_0)}{k - u_0 + ap'(h)} \geqslant \frac{a\sqrt{h}p'(h)}{a\sqrt{p'(h)} + \sqrt{a}}, \\ k - u_0 = k - u_1 + (u_1 - u_0) \leqslant \sqrt{hp'(h)} + \frac{h}{a}. \end{gathered} \tag{3.11}$$

Thus, inequalities (3.7) follow from (3.8) and (3.11).

The use of the *a priori* estimates (3.7) and Eqs. (3.3) makes it possible to show that the functions u, u_λ, H and k are restricted in an arbitrary segment $h \in [\sigma, b]$ ($0 < \sigma < h_0 < b$). In this case, the inequalities $|u - k| > \varepsilon(\sigma, b, \|\boldsymbol{V}_0\|)$ and $|H| > \varepsilon(\sigma, b, \|\boldsymbol{V}_0\|)$ are valid. The theorem may be proved by repeatedly applying the local theorem about the existence of a solution of problem (3.3). Thus, Theorem 2 is proved.

The last step in constructing the simple wave is obtaining the solution of the Cauchy problem $h_t + k(h)h_x = 0$, $h|_{t=0} = \tau(x)$. As a result, we have a pair of functions $u(h(t,x),\lambda)$ and $H(h(t,x),\lambda)$ that satisfy system (1.7).

Remark. Particular solutions of Eqs. (1.4) of the form

$$u = u(\alpha(t,x), y), \quad v = \alpha_x \tilde{v}(\alpha(t,x), y), \quad h = h(\alpha(t,x)) \tag{3.12}$$

are called simple waves.

4. Symmetries and Invariant Solutions. We build the exact solutions under the assumption that the excess pressure in the fluid is related to the cross-sectional area of the tube by a power law, i.e.,

$$p(h) = C_1 h^\gamma + C_2 \qquad (\gamma \neq 0), \tag{4.1}$$

where C_1, C_2, and γ are some constants. Exactly this approximation (where h varies in proportion to S) is very often used to close the equations of motion of a fluid in elastic tubes. Following [10], in order to simplify the boundary conditions in (1.4), we introduce the variable

$$z = \frac{y}{h(t,x)}, \qquad w = \frac{dz}{dt} = \frac{1}{h}\Big(v - \frac{y}{h}(h_t + uh_x)\Big). \tag{4.2}$$

With due regard for (1.4), (4.2), and (4.1), for the functions $u(t,x,z)$, $w(t,x,z)$, and $\eta(t,x) = C_1 h^\gamma$, we obtain the equations

$$u_t + uu_x + wu_z + \eta_x = 0, \quad \eta_t + u\eta_x + \gamma\eta(u_x + w_z) = 0, \quad w|_{z=0} = w|_{z=1} = 0. \tag{4.3}$$

Let us derive the group of transformations admitted by Eqs. (4.3). To this end, we add the equation $\eta_z = 0$ to system (4.3) and apply the first extension of the operator $X = \xi^t\partial_t + \xi^x\partial_x + \xi^z\partial_z + \zeta^u\partial_u + \zeta^w\partial_w + \zeta^\eta\partial_\eta$ to this system (ξ^i and ζ^j depend on the variables t, x, z, u, w, and η). After the transformations, the system of determining equations acquires the form

$$\begin{gathered} \xi^t_{tt} = \xi^x_{tx}, \qquad \xi^x_{tt} = \xi^x_{xx} = 0, \qquad \xi^z_{tz} = (2/\gamma - 1)\xi^x_{tx}, \qquad \xi^z_{xz} = \xi^z_{zz} = 0, \\ \zeta^u = (\xi^x_x - \xi^t_t)u + \xi^x_t, \qquad \zeta^w = (\xi^z_z - \xi^t_t)w + \xi^z_t + u\xi^z_x, \qquad \zeta^\eta = 2(\xi^x_x - \xi^t_t)\eta, \end{gathered} \tag{4.4}$$

where $\xi^t = \xi^t(t)$, $\xi^x = \xi^x(t,x)$, and $\xi^z = \xi^z(t,x,z)$. From these equations, it follows that ξ^z is a function that is linear with respect to the variable z. The use of the boundary conditions in (4.3) results in relations $\xi^z|_{z=0} = 0$ and $\xi^z|_{z=1} = 0$; hence, $\xi^z \equiv 0$. Integrating Eqs. (4.4) (for $\gamma \neq 2$), we find the Lee algebra L_5 of admissible operators: $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = t\partial_x + \partial_u$, $X_4 = t\partial_t + x\partial_x - w\partial_w$, and $X_5 = x\partial_x + u\partial_u + 2\eta\partial_\eta$. If $\gamma = 2$, the operator $X_6 = t^2\partial_t + tx\partial_x + (x - tu)\partial_u - 2tw\partial_w - 2t\eta\partial_\eta$ arises in this consideration.

Below, all representatives of the optimum system of subalgebras of the Lee algebra L_5 of rank one are listed [12]: 1) X_1; 2) X_2; 3) X_3; 4) $X_1 + X_3$; 5) X_4; 6) $X_3 + X_4$; 7) $X_1 + X_5$; 8) $X_2 - X_4 + X_5$; 9) $\alpha X_4 + X_5$.

Submodels. For representatives of the optimum system of rank one, given below are the set of basis invariants J, the representation of the solution, and the factor-system E/H [$H(\alpha_i X_i)$ denotes a subalgebra].

1) $H(X_1)$ and $J = (x, z, u, w, \eta)$. The solution representation is

$$u = u(x,z), \qquad w = w(x,z), \qquad \eta = \eta(x)$$

and the factor-system E/H is

$$uu_x + wu_z + \eta_x = 0, \qquad u\eta_x + \gamma\eta(u_x + w_z) = 0, \qquad w|_{z=0} = w|_{z=1} = 0. \tag{4.5}$$

2) $H(X_2)$ and $J = (t, z, u, w, \eta)$. The solution representation is

$$u = u(t,z), \qquad w = w(t,z), \qquad h = h(t)$$

and the factor-system E/H is

$$u_t + wu_z = 0, \qquad \eta_t + \gamma\eta w_z = 0, \qquad w|_{z=0} = w|_{z=1} = 0. \tag{4.6}$$

3) $H(X_3)$ and $J = (t, z, tu - x, w, \eta)$. The solution representation is

$$u = (\varphi(t,z) + x)t^{-1}, \qquad w = w(t,z), \qquad \eta = \eta(t)$$

and the factor-system E/H is

$$\varphi_t + w\varphi_z = 0, \qquad \eta_t + \gamma\eta(t^{-1} + w_z) = 0, \qquad w|_{z=0} = w|_{z=1} = 0. \tag{4.7}$$

4) $H(X_1 + X_3)$ and $J = (x - t^2/2, z, u - t, w, \eta)$. The solution representation is

$$\xi = x - t^2/2, \qquad u = \varphi(\xi, z) + t, \qquad w = w(\xi, z), \qquad \eta = \eta(\xi)$$

and the factor-system E/H is

$$1 + \varphi\varphi_\xi + w\varphi_z + \eta_\xi = 0, \qquad \varphi\eta_\xi + \gamma\eta(\varphi_\xi + w_z) = 0, \qquad w|_{z=0} = w|_{z=1} = 0. \tag{4.8}$$

5) $H(X_4)$ and $J = (xt^{-1}, z, u, tw, \eta)$. The solution representation is

$$\xi = xt^{-1}, \qquad u = u(\xi, z), \qquad w = \varphi(\xi, z)t^{-1}, \qquad \eta = \eta(\xi)$$

and the factor-system E/H is

$$(u - \xi)u_\xi + \varphi u_z + \eta_\xi = 0, \qquad (u - \xi)\eta_\xi + \gamma\eta(u_\xi + \varphi_z) = 0, \qquad \varphi|_{z=0} = \varphi|_{z=1} = 0. \tag{4.9}$$

6) $H(X_3 + X_4)$ and $J = (xt^{-1} - \ln t, z, u - \ln t, tw, \eta)$. The solution representation is

$$\xi = xt^{-1} - \ln t, \qquad u = \varphi(\xi, z) + \ln t, \qquad w = \psi(\xi, z)t^{-1}, \qquad \eta = \eta(\xi)$$

and the factor-system E/H is

$$(\varphi - \xi - 1)\varphi_\xi + \psi\varphi_z + \eta_\xi + 1 = 0,$$

$$(\varphi - \xi - 1)\eta_\xi + \gamma\eta(\varphi_\xi + \psi_z) = 0, \qquad \psi|_{z=0} = \psi|_{z=1} = 0.$$

7) $H(X_1 + X_5)$ and $J = (x\exp(-t), z, u\exp(-t), w, \eta\exp(-2t))$. The solution representation is

$$\xi = x\exp(-t), \qquad u = \varphi(\xi, z)\exp(t), \qquad w = w(\xi, z), \qquad \eta = s(\xi)\exp(2t)$$

and the factor-system E/H is

$$\varphi + (\varphi - \xi)\varphi_\xi + w\varphi_z + s_\xi = 0, \qquad 2s + (\varphi - \xi)s_\xi + \gamma s(\varphi_\xi + w_z) = 0, \qquad w|_{z=0} = w|_{z=1} = 0.$$

8) $H(X_2 - X_4 + X_5)$ and $J = (t\exp(x), z, tu, tw, t^2\eta)$. The solution representation is

$$\xi = t\exp(x), \qquad u = t^{-1}\varphi(\xi, z), \qquad w = t^{-1}\psi(\xi, z), \qquad \eta = t^{-2}s(\xi)$$

and the factor-system E/H is

$$-\varphi + \xi(\varphi + 1)\varphi_\xi + \psi\varphi_z + \xi s_\xi = 0, \qquad -2s + \xi(\varphi + 1)s_\xi + \gamma s(\xi\varphi_\xi + \psi_z) = 0, \qquad \psi|_{z=0} = \psi|_{z=1} = 0.$$

9) $H(\alpha X_1 + X_5)$ and $J = (t^{-(\beta+1)}x, z, t^{-\beta}u, tw, t^{-2\beta}\eta)$ $(\beta = \alpha^{-1})$. The solution representation is

$$\xi = t^{-(\beta+1)}x, \qquad u = t^{2\beta}\varphi(\xi, z), \qquad w = t^{-1}\psi(\xi, z), \qquad \eta = t^{2\beta}s(\xi)$$

and the factor-system E/H is

$$\begin{gathered}(\varphi - (1 + \beta)\xi)\varphi_\xi + \psi\varphi_z + s_\xi + \beta\varphi = 0,\\ (\varphi - (1 + \beta)\xi)s_\xi + \gamma s(\varphi_\xi + \psi_z) + 2\beta s = 0, \qquad \psi|_{z=0} = \psi|_{z=1} = 0;\end{gathered} \tag{4.10}$$

For $\alpha = 0$, we have $J = (t, z, x^{-1}u, w, x^{-2}\eta)$; then, the solution representation is

$$u = x\varphi(t, z), \qquad w = w(t, z) \qquad \eta = x^2 s(t).$$

and the factor-system E/H is

$$\varphi_t + w\varphi_z + \varphi^2 + 2s = 0, \qquad s_t + \gamma s w_z + (2 + \gamma)\varphi s = 0, \qquad w|_{z=0} = w|_{z=1} = 0. \tag{4.11}$$

Invariant Solutions. Submodel (4.6). The solution $u = u(z)$, $w = 0$, $\eta = \eta_0$ describes the shear flows of the fluid [$u = u(y)$, $v = 0$, $h = \text{const}$].

Submodel (4.7). Integration of Eqs. (4.7) and inversion of substitution (4.2) yields the solution of Eqs. (1.4) $u = (\varphi(tx) + x)/t$, $v = -y/t$, $h = c/t$ ($c = \text{const}$) that describes the deformation of the channel by compressing pressure [12].

Submodel (4.8). The solutions invariant with respect to translations in time and with respect to Galilean transformations can be conveniently found in Eulerian–Lagrangian variables. Indeed, the factor-system for Eqs. (1.7) has the form

$$\left(\frac{\varphi^2}{2} + p\left(\int_0^1 H\,d\lambda\right) + \xi\right)_\xi = 0, \qquad (\varphi H)_\xi = 0,$$

where $\varphi(\xi, \lambda) = u - t$ and $\xi = x - t^2/2$. From here, we have $u = t \pm \sqrt{2(C(\lambda) - \xi - p(h))}$ and $H = (u - t)^{-1}$; the function $h(\xi)$ is given by the equation

$$h - \int_0^1 \frac{d\lambda}{\sqrt{2(C(\lambda) - \xi - p(h))}} = 0.$$

Such solutions, which describe flows with a critical layer ($u - t$ turns into zero), are analogous to those obtained in [12] for the equations of eddying shallow water.

Submodel (4.5). The stationary solutions may be conveniently constructed using the equations of motion in the form (1.7). By analogy with submodel (4.8), integration of the equations reduces to determining the function $h(x)$ from a finite relation. Such a solution [with $p(h) = h$] was examined in [11].

Submodel (4.10). Equations (4.10) admit the extension $\xi \to a\xi$, $\varphi \to a\varphi$, $s \to a^2 s$. The invariant solution arising in this submodel has the form

$$\varphi = \xi A(z), \qquad \psi = B(z), \qquad s = s_0 \xi^2. \tag{4.12}$$

For the functions $A(z)$ and $B(z)$, we have the equations

$$A^2 - A + A'B + 2s_0 = 0, \qquad (2 + \gamma)A - 2 + \gamma B' = 0, \qquad B(0) = B(1) = 0. \tag{4.13}$$

Eliminating the function $B(z)$ from system (4.13), we obtain

$$B(z) = \frac{A(z) - A^2(z) - 2s_0}{A'(z)}. \tag{4.14}$$

In this case, the function $A(z)$ should satisfy the equation

$$(A^2 - A + 2s_0)\frac{d^2 A}{dz^2} + \Big(\frac{2}{\gamma} - 1\Big)(A - 1)\Big(\frac{dA}{dz}\Big)^2 = 0.$$

Introducing $\lambda_1 = (1 - \sqrt{1 - 8s_0}\,)/2$ and $\lambda_2 = (1 + \sqrt{1 - 8s_0}\,)/2$ and interchanging the variables A and z, we have

$$(A - \lambda_1)(A - \lambda_2)\frac{d^2 z}{dA^2} + \Big(\frac{2}{\gamma} - 1\Big)\Big(A - 1\Big)\frac{dz}{dA} = 0.$$

In view of (4.14), the boundary conditions are fulfilled if $z = 0$ for $A = \lambda_2$ and $z = 1$ for $A = \lambda_1$. A new variable $\tau = (\lambda_2 - A)(\lambda_2 - \lambda_1)^{-1}$ may be conveniently introduced. The function $z(\tau)$ satisfies the equation

$$\tau(1 - \tau)\frac{d^2 z}{d\tau^2} + \Big(\frac{2}{\gamma} - 1\Big)\Big(\tau + \frac{\lambda_1}{\lambda_2 - \lambda_1}\Big)\frac{dz}{d\tau} = 0. \tag{4.15}$$

The solution of Eq. (4.15) is given by an incomplete beta-function $\beta(a, b, \tau) = \int_0^\tau t^{a-1}(1 - t)^{b-1}\,dt$ ($a > 0$, $b > 0$). Returning to the variable A and satisfying the boundary conditions, we obtain

$$z(A) = \frac{\beta(a, b, (\lambda_2 - A)/(\lambda_2 - \lambda_1))}{\beta(a, b, 1)}, \qquad a = \frac{\gamma - \lambda_1(\gamma + 2)}{\gamma(\lambda_2 - \lambda_1)}, \qquad b = \frac{2 - \lambda_1(\gamma + 2)}{\gamma(\lambda_2 - \lambda_1)}. \tag{4.16}$$

Conditions $a > 0$ and $b > 0$ are fulfilled if

$$\gamma > 0, \qquad 0 < s_0 < \frac{1}{8}\Big(1 - \Big(\frac{2 - \gamma}{2 + \gamma}\Big)^2\Big).$$

Thus, formulas (4.16), (4.14), and (4.12) yield the solution of submodel (4.10).

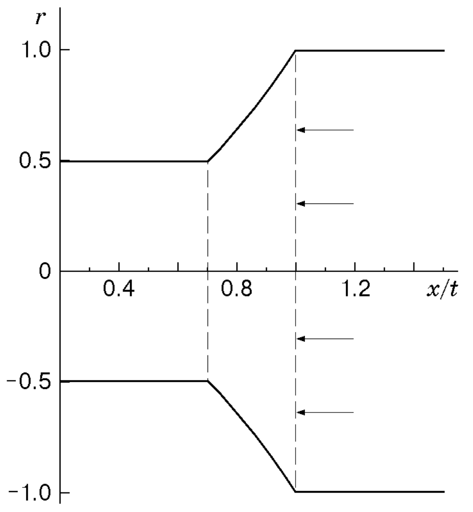

Fig. 1

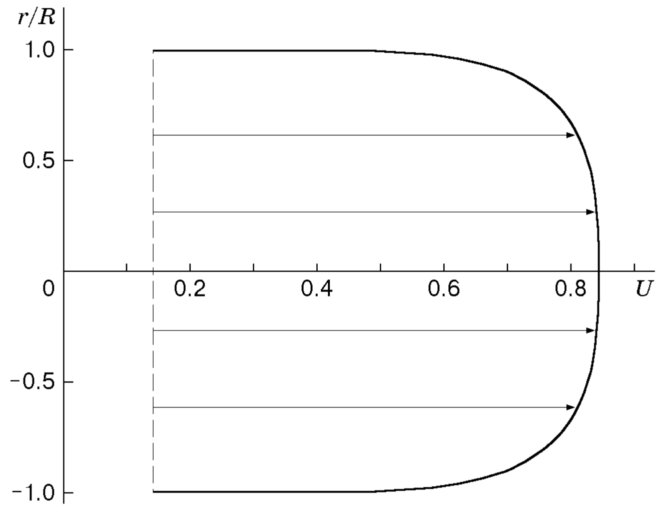

Fig. 2

Submodel (4.9). By analogy with model (4.10), we seek the solution of Eqs. (4.9) in the form

$$u = \xi A(z), \qquad \varphi = B(z), \qquad \eta = s_0\xi^2. \tag{4.17}$$

Then, for the functions $A(z)$ and $B(z)$, we have system (4.13); hence, a particular solution of submodel (4.9) is given by formulas (4.17), (4.16), and (4.14).

Submodel (4.11). Factor-system (4.11) is invariant with respect to the extension $t \to a^{-1}t$, $\varphi \to a\varphi$, $w \to aw$, $s \to a^2 s$; hence, its solution may be represented as

$$\varphi = t^{-1}A(z), \qquad w = t^{-1}B(z), \qquad s = s_0 t^{-2}. \tag{4.18}$$

The functions $A(z)$ and $B(z)$ satisfy Eqs. (4.13), and a particular solution of the submodel is given by formulas (4.18), (4.16), and (4.14).

Submodel (4.8). Equations (4.8) admit the extension $\xi \to a^2\xi$, $\varphi \to a\varphi$, $w \to a^{-1}w$, $\eta \to a^2\eta$. The latter allows us to seek a particular solution of this model which has the form

$$\varphi = |\xi|^{1/2}A(z), \qquad w = |\xi|^{-1/2}B(z), \qquad \eta = \eta_0\xi. \tag{4.19}$$

We assume that $\xi < 0$ and $-1 < \eta_0 < 0$. Substituting (4.19) into (4.8), we obtain the following equations for the functions $A(z)$ and $B(z)$:

$$-A^2/2 + A'B + 1 + \eta_0 = 0, \qquad (1/2 + 2/\gamma)A - B' = 0, \qquad B(0) = B(1) = 0.$$

Let us express $B(z)$ from the first equation of this system

$$B = (A^2/2 - 1 - \eta_0)/A'^2; \tag{4.20}$$

insert $B'(z)$ into the second equation, and interchange the variables A and z. Then, for the function $z(A)$, we have the equation

$$(\mu^2 - A^2)\frac{d^2z}{dA^2} + A\Big(\frac{4}{\gamma} - 1\Big)\frac{dz}{dA} = 0, \tag{4.21}$$

in which $\mu = \sqrt{2(1+\eta_0)}$. To meet the boundary conditions, we have to demand that the variable z be vanishing (for $A = \mu$) or turning into unity (for $A = -\mu$); hence, the solution of (4.21) of interest has the form

$$z(A) = \frac{\beta(2/\gamma + 1/2, 2/\gamma + 1/2, (\mu - A)/(2\mu))}{\beta(2/\gamma + 1/2, 2/\gamma + 1/2)} \qquad (\gamma > 0, \quad \gamma < -4). \tag{4.22}$$

Formulas (4.22), (4.20), and (4.18) give a particular solution of submodel (4.11).

Characteristic Flow Modes. The solutions of submodels (4.8)–(4.11) constructed above belong to the class of simple waves (3.12). We consider in more detail the solution of submodel (4.9) (solutions of other submodels are analogous to this solution). According to (4.17), (4.1), and (4.2), the solution of system (1.4) has the form

$$u = \frac{x}{t} A\Big(\frac{y}{h}\Big), \ \ v = \frac{h}{t} B\Big(\frac{y}{h}\Big) + \frac{2}{\gamma t}\Big(\frac{s_0}{C_1}\Big)^{1/\gamma}\Big(\frac{x}{t}\Big)^{2/\gamma}\Big(A\Big(\frac{y}{h}\Big) - 1\Big), \ \ h = \Big(\frac{s_0}{C_1}\Big)^{1/\gamma}\Big(\frac{x}{t}\Big)^{2/\gamma}, \tag{4.23}$$

where the functions A and B are defined by formulas (4.16) and (4.14), respectively. The value of $k = -h_t/h_x = x/t$ coincides with the larger root of the characteristic equation (2.4) calculated from solution (4.23); hence, $u - k < 0$ and, as the simple wave propagates across the shear flow, in which $u = u_0(y)$, $v = 0$, and $h = h_0$, the fluid particles enter the region of the simple wave from the right. Let $\gamma = 1/2$, $C_1 = s_0$, and the attachment of the simple wave to the shear flow occurs along the characteristic $x/t = \text{const} = 1$. In the region occupied by the simple wave, both the excess pressure and in the flow acceleration decrease (relative to the wave). In this situation, the cross-sectional area of the elastic tube decreases (Fig. 1). Let the excess pressure vanish at $x/t = \xi_*$; then, the downstream flow is a shear flow attached to the simple wave along the characteristic $x/t = \xi_*$. The profile of the axial velocity U as a function of the ratio between the radial coordinate r and the dimensionless radius of the tube R at $x/t = 1$ is shown in Fig. 2. For other values of $\gamma > 0$ and s_0/C_1, the behavior of the flow is qualitatively similar.

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REFERENCES

1. T. Pedly, *The Fluid Mechanics of Large Blood Vessels*, Cambridge Univ. Press (1980).
2. A. N. Volobuev, "Fluid flow in tubes with elastic walls," *Usp. Fiz. Nauk*, **165**, No. 2, 177–186 (1995).
3. V. M. Teshukov, "Hyperbolicity of the equations of motion of a long wave," *Dokl. Akad. Nauk SSSR*, **284**, No. 3, 555–562 (1985).
4. L. V. Ovsyannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).
5. V. M. Teshukov, "Long-wave approximation model for a gas shear flow in a channel of varying area," *Prikl. Mekh. Tekh. Fiz.*, **39**, No. 1, 15–27 (1998).
6. B. J. Benney, "Some properties of long nonlinear waves," *Stud. Appl. Math.*, **52**, 45–50 (1973).
7. V. E. Zakharov, "Benney's equations and the quasi-classic approximation in the inverse-problem method," *Funkts. Anal. Prilozhen.*, **14**, No. 2, 15–24 (1980).
8. V. M. Teshukov, "Long waves in an eddying barotropic fluid," *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 6, 17–26 (1994).
9. V. M. Teshukov, "Simple waves on a free-boundary flow of an ideal incompressible fluid," *Prikl. Mekh. Tekh. Fiz.*, **38**, No. 2, 48–57 (1997).
10. P. L. Sachdev and Ph. Varugheze, "Invariance group properties and exact solutions of equations describing time-dependent free surface flows under gravity," *Quart. Appl. Math.*, **43**, No. 2, 465–482 (1986).
11. E. Varley and P. A. Blythe, "Long eddies in sheared flows," *Stud. Appl. Math.*, **68**, 103–187 (1983).
12. A. A. Chesnokov, "Exact solutions of the equations of vortex shallow water," *Prikl. Mekh. Tekh. Fiz.*, **38**, No. 5, 44–55 (1997).
13. A. A. Chesnokov, "Vortex motion of fluid in a narrow channel," *Prikl. Mekh. Tekh. Fiz.*, **39**, No. 4. 38–49 (1998).
14. Yu. L. Daletskii and M. G. Krein, *Stability of Solutions of Banach-Space Differential Equations* [in Russian], Nauka, Moscow (1970).